

# An Elliptic PDE Approach for Shape Characterization

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**Abstract**—This paper presents a novel approach to analyze the shape of anatomical structures. Our methodology is rooted in classical physics and in particular Poisson's equation, a fundamental partial differential equation [1]. The solution to this equation and more specifically its equipotential surfaces display properties that are useful for shape analysis. We present a numerical algorithm to calculate the length of streamlines formed by the gradient field of the solution to this equation for 2D and 3D objects. The length of the streamlines along the equipotential surfaces was used to build a new function which can characterize the shape of objects. We illustrate our method on 2D synthetic and natural shapes as well as 3D medical data.

**Keywords**—Shape Analysis, Partial Differential Equation

## I. INTRODUCTION

Shape analysis methods play a key role in the study of medical images. They allow going beyond simple volumetric measures and providing a more intuitive idea of the changes an anatomical structure undergoes. There are mainly three classes of shape analysis methods. The first class relies on a feature vector as a representation of shape. Spherical harmonics or invariant moments have been used in this context [2, 3]. Classes of shapes can then be discriminated using clustering methods such as principal component analysis. These methods are usually numerically stable and relevant statistics can be computed from them. However, their interpretation is often difficult and they rarely provide an intuitive description of the shape. The second class of methods is based on a surface boundary representation of the object and the study of the mechanical deformations required to transform one object into another [4, 5]. This popular technique is very intuitive, but relies on registration methods which are difficult to implement and not always reliable. Calculating significant statistics from the deformation also poses a challenge. The third class makes use of medial representations which provide insightful information on the symmetry of the object. Unfortunately, the medial models still need to be registered with each other before any statistics can be derived [6-8]. In clinical studies, different classes of methods are often combined in order to obtain intuitive and statistically significant results, see for example [9].

In this paper, we propose a novel shape analysis method based on Poisson's equation with a Dirichlet boundary condition. This equation, most known in electrostatics, has very interesting properties for the study of shape. Most notably, its solution is always smooth, has one sink point and can be made independent of the scale of the original object. Our approach is to extract a curve which expresses the relationship between displacement and energy level. We name this curve the shape characteristic and argue that it is a very useful tool for shape analysis and classification. The next section provides details on Poisson's equation and how the displacement function can be computed. Section III illustrates the method on synthetic and natural 2D objects as well as 3D medical data.

## II. METHODOLOGY

### A. Poisson's Equation

Poisson's equation is fundamental to mathematical physics and has been widely used over a range of phenomena. Examples include electrostatic fields, gravitational fields, thermo-dynamic flows and other applications. Mathematically, Poisson's equation is a second-order elliptic partial differential equation defined as:

$$\Delta u = -1 \quad (1)$$

Poisson's equation is independent of the coordinate system and characterizes the entire domain (volume in 3D) not only its boundary. Functions  $u$  satisfying Poisson's equation are called potential functions. These functions have many mathematical properties related to the underlying geometry of the structure. Among the properties are the following:

1. The shape of the potential function is correlated to the geometry of the structure. This correlation gives a mathematical meaning to the medical concept of anatomical sublayers.
2. In the special case of Dirichlet conditions on the outer surface of a closed homogeneous domain, the potential function converges smoothly to a single sink point. Moreover, a unique streamline can be drawn from each point of the boundary to the sink point by following the gradient field of the potential function.

3. The pattern of streamlines is independent of the value of the potential on the boundary and is closely related to the shape of the domain.

The length of each streamline can be calculated by summing the Euclidean distances between neighboring points along the streamline. For example, in electrostatics, this length is called the ‘electric displacement’.

In Figure 1, we illustrate the solution of Poisson’s equation for simple two dimensional domains, a circle (Figure 1a) and a square (Figure 1b). In both examples, the initial conditions on the boundary were set to  $u = 100$ . The solution represents a layered set of equipotential curves making a smooth transition from the outer contour to the center. The equipotential curves for a circle are simply smaller circles. However, the equipotential curves inside a square smoothly change shape as they approach the sink point. This change is illustrated in Figure 1a where two streamlines connect the sink point to two equipotential points inside the circle. Figure 1b shows similar streamlines inside the square. Inside a circle the ‘electric displacement’ along an equipotential curve is constant. Inside a square, this displacement varies due to the shape of the boundary. Note however, that the variation of displacement decreases from one equipotential level to another while moving towards the sink point. We used this concept of displacement as the basis of our approach to apply Poisson’s equation for the analysis of shape of anatomical structures.

### B. Calculating the displacement

Since there are many standard numerical methods for solving Poisson’s equation (1), we will not discuss in this work the numerical solution of Poisson’s equation for 3D MRI. We refer the reader to [10] for details on numerical solutions of standard partial differential equations. We will focus instead on the computation of the displacement for each point in the given domain.

Because all streamlines converge to one unique sink point,  $P_s$ , we can design a downwinding algorithm to calculate the displacement at a voxel  $P_0$  by summation of the Euclidean distances between consecutive voxels along the streamline connecting  $P_0$  to  $P_s$ .

First,  $P_s$  is found by solving the following equation:

$$\|\text{grad}(u)\| = 0 \quad (2)$$

The displacement  $D$  at a voxel  $P_0$  is then defined as:

$$D(P_0) = \sum L_i \quad (3)$$

$L_i$  is the Euclidean distance between consecutive voxels  $P_i$  and  $P_{i+1}$  on the streamline connecting  $P_0$  to the sink point. Depending on the direction of the gradient field at voxel  $P_i$ ,  $L_i$  can have one of the following values:

$h_1, h_2, h_3, (h_1^2 + h_2^2)^{1/2}, (h_1^2 + h_3^2)^{1/2}, (h_2^2 + h_3^2)^{1/2}$  or  $(h_1^2 + h_2^2 + h_3^2)^{1/2}$ ;

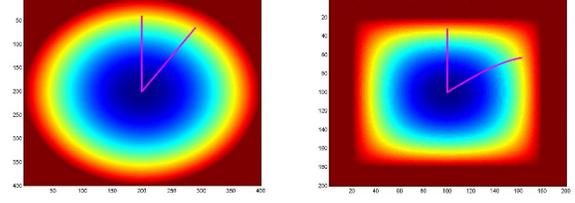


Fig. 1. a) potential function inside a circle with streamlines of two equipotential points. b) potential function inside a square with streamlines of two equipotential points.

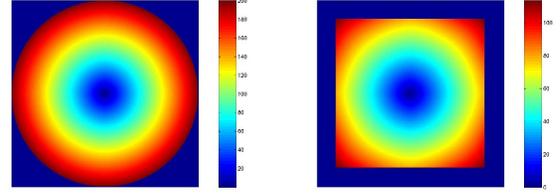


Fig. 2. Displacement map of a circle (left) and a square (right).

$h_1$  is the voxel width,  $h_2$  the voxel height and  $h_3$  the slice thickness.

The direction of the gradient is assessed using a simple transformation from a Cartesian to a spherical coordinate system. The azimuth  $\theta$  and the elevation  $\phi$  of the gradient are calculated at  $P_i$  and used to determine  $L_i$ . The coordinates of the following voxel on the streamline,  $P_{i+1}$ , can be computed using;

$$\begin{cases} x_{i+1} = x_i + L_i \cdot \sin\phi_i \cdot \cos\theta_i; \\ y_{i+1} = y_i + L_i \cdot \sin\phi_i \cdot \sin\theta_i; \\ z_{i+1} = z_i + L_i \cdot \cos\phi_i \end{cases} \quad (4)$$

$L_i$  is added to the current  $D(P_0)$  using (3), the procedure is then repeated at  $P_{i+1}$  until the sink point  $P_s$  is reached. Figure 2 presents the displacement maps of a circle and square.

### C. Analysis of shape using Poisson’s equation

Earlier, we demonstrated that the dynamics of change of the equipotential surfaces inside the domain while approaching the center is related to the domain geometry. To evaluate this process we first define the normalized drop of potential  $E$  at an equipotential surface  $S_i$  as:

$$E = (U_0 - U_i)/(U_0 - U_s); \quad (5)$$

$U_0, U_s$  and  $U_i$  are the potentials on the boundary, at the sink point and on the current equipotential surface respectively.  $E$  characterizes the amount of energy needed to transform the surface boundary into the current equipotential surface.

We then introduce  $v$ , the coefficient of variance of the displacement along the current equipotential surface  $S_i$ :

$$v(E) = \text{stdev}(D(S_i))/\text{mean}(D(S_i)) \quad (6)$$

Function  $v(E)$ , which we call the ‘‘shape characteristic’’, displays some very interesting properties:

1. It is independent of the potential on the outer boundary.
2. It is independent of the overall volume and defined exclusively by the shape.
3. A given value  $v$  corresponds to a known drop of potential  $E$ .

Thus, the shape of a structure can be represented by the “shape characteristic” which expresses the relationship between the variation of the displacement and the energy drop and is independent of overall size and initial conditions on the outer boundary of the object.

### III. RESULTS

#### A. Shape Characteristic for 2D objects.

In this section, we illustrate how the shape characteristic is connected to the shape of the object. First, in fig. 3, we present three isosceles triangles each with a different acute angle (30,45 and 60 degrees). The shape characteristic is almost linear and the slope of the line is nearly identical for the three triangles.

Next, we computed the shape characteristic for natural shapes. Fig. 4 presents the analysis of three different images of hands and Fig. 5 shows our results for three different horses. One can see that the shape characteristics are very similar for classes of similar objects.

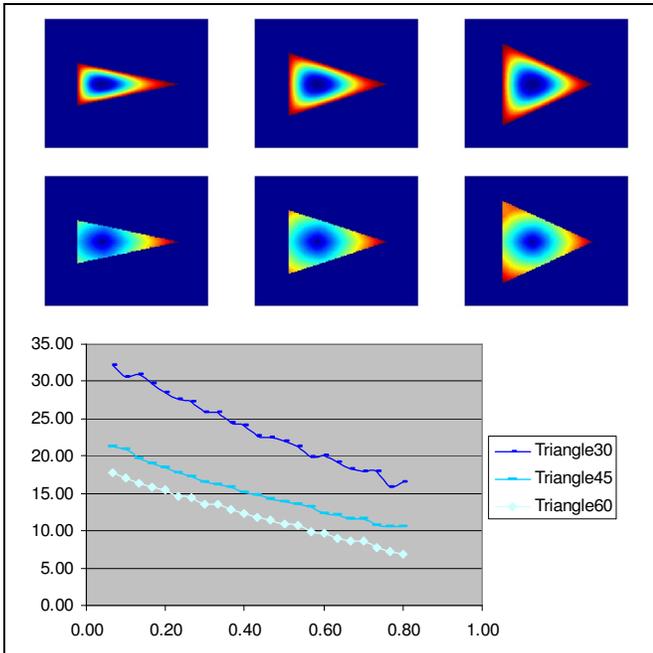


Fig. 3 Three isosceles triangles with different acute angles display very similar shape characteristics. Top: Solution to Poisson’s equation, Middle: Displacement map, Bottom: Shape characteristics.

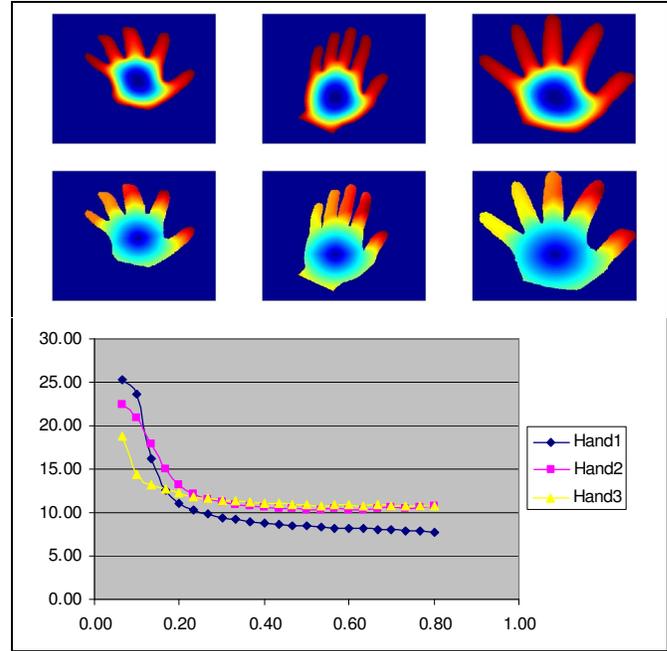


Fig. 4 Shape characteristics of three different hands. Top: Solution to Poisson’s equation, Middle: Displacement map, Bottom: Shape characteristics.

For the hands,  $v(E)$  first drops significantly and then becomes almost constant. On the other hand, the shape characteristic of a horse typically increases at a nearly constant rate.

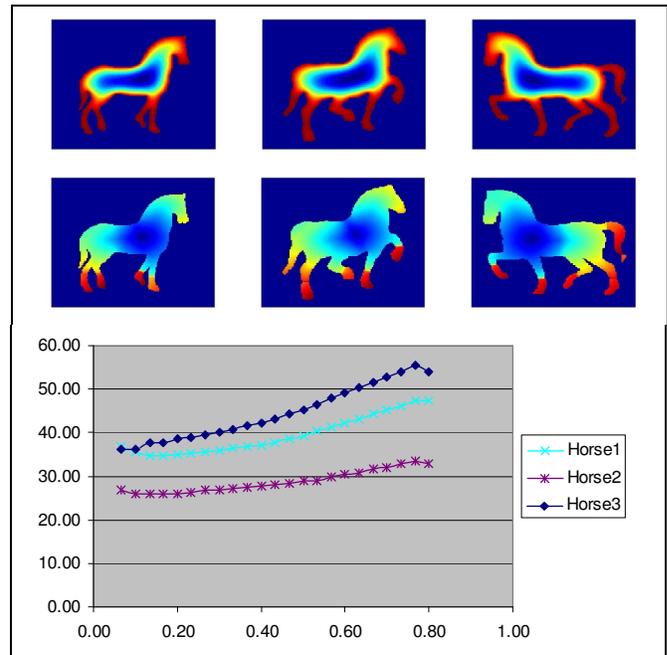


Fig. 5 Shape characteristics of three different horses. Top: Solution to Poisson’s equation, Middle: Displacement map, Bottom: Shape characteristics.

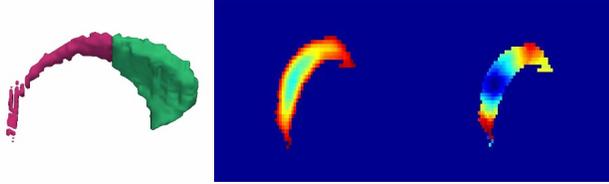


Fig. 6 3D rendering of the caudate nucleus (left), sagittal cross section displaying the potential function (middle) and displacement map (right).

These results suggest that the shape characteristic can be used to describe cluster different objects into classes of shapes. Standard curve analysis techniques such as looking at the curvature could also, in theory, be used to further parcellate clusters into subclasses. For example, one could analyse the “hand” class and distinguish between long fingers and short fingers hands.

### B. Medical Data

Finally, we present some preliminary results on medical data. An MRI image of the brain was acquired with a 1.5T GE scanner, using a SPoiled Gradient Recalled (SPGR) sequence yielding to a (256x256x124) volume with (0.9375x0.9375x1.5mm) voxel dimensions. The scans were acquired coronally. The caudate nucleus, an essential part in the “cognitive” circuitry connecting the frontal lobe to subcortical structures of the brain, was drawn manually and separated by an Anterior/Posterior boundary. We then applied our algorithm to compute the potential and displacement maps. The results are presented in fig. 6, for clarity of presentation only cross sections of the 3D potential map and displacement map are shown. We are currently working on measuring shape differences of the caudate between normal controls and patients diagnosed with schizophrenia. We believe the ‘shape characteristic’ will be an essential tool for this task.

## V. CONCLUSION

We have developed a novel method for shape analysis of anatomical structures. Our method is based on using the solution of Poisson’s equation to assess the dynamics of change of shape of the equipotential surfaces inside the structure. We have developed an algorithm for calculating the displacement maps defined by the length of the streamlines generated by the gradient field of the potential function. We used these maps to introduce a new function, called ‘shape characteristic’, which characterizes in a unique way the shape of a structure. Our method was validated on synthetic and natural shapes in both 2D and 3D. We believe that certain features derived from Poisson’s equation can be very useful for the analysis of shape. Our first attempt at describing the shape of an object with the shape characteristic displays some interesting properties and

suggests that it could be an efficient tool for shape classification and clustering. It seems also natural to attempt to discriminate similar structures with such a tool. For example one could look at a population of anatomical structure such as the caudate and try to correlate shape characteristic properties with different factors such as aging or diseases.

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